Indian Statistical Institute
Bangalore Centre
B.Math (Hons.) III Year 2015-2016
First Semester
Statistics III

Semestral Examination   Date : 9.11.16

Time : 3 hours and 30 minutes.

Answer as many questions as possible. The maximum you can score is 120.
All symbols have their usual meaning, unless stated otherwise.
State clearly the results you use.

Notation : a^{ij} is the (i, j)th element of A^{-1}.

1. Consider a random vector $X = (X_1, \cdots X_p)'$ having covariance matrix Σ. Define Partial correlation coefficient $\rho_{12.3\cdots p}$ and explain what it says about X_1 and X_2. Show that

$$\rho_{12.3\cdots p} = -\sigma^{12}/(\sigma^{11}\sigma^{22}).$$

[2 + 3 + 8 = 13]

2. Consider the model

$$y_{ij} = \mu + \alpha_i + \beta_j + \varepsilon_{ij},\ j = 1, 2, ..., b,\ i = 1, 2...a$$

   (a) For each of the following parametric functions, either prove that it is not estimable or provide two distinct unbiased estimators of which one must be the BLUE. Justify your answer. [Assume $b \geq 3$].
   (i) μ, (ii) α_1, (iii) $3\beta_1 - 2\beta_2 - \beta_3$.
   (b) Write down the model in the form $Y = X_0\ \mu + X_1\ \alpha + X_2\ \beta + \varepsilon$, describing the matrices X_0, X_1, X_2.
   (c) Show that the rank of $X = [X_0|X_1|X_2]$ is $a + b - 1$

[(3 + 3 + 5) + 3 + 5 = 19]

3. Suppose $X = (X_1, \cdots X_p)' \sim N_p(0, I_p)$. Suppose $X'AX \sim \chi^2(a)$ and $X'BX \sim \chi^2(b)$, where $A - B$ is nonnegative definite. Show that $X'(A - B)X \sim \chi^2(a - b)$. [8]

4. Consider a $p \times p$ random matrix S which is positive definite a.s. If $S \sim W_p(n, \Sigma)$, where Σ is positive definite, prove the following results.
   (a) $Z = \sigma^{pp}/s^{pp} \sim \chi^2(n - p + 1)$.
   (b) Z is independent of $((s_{ij}))_{1 \leq i,j \leq p-1}$.
   (c) $det(S)/det(\Sigma)$ is distributed as the product of p independent χ^2 variables with degrees of freedom $n - p + 1, \cdots n - 1, n$.

[12 + 2 + 8 = 22]

5. (a) Suppose V and A are $p \times p$ p.d. matrices and $H(V) = t \log \det V - tr(VA)$. Then, show that

$$sup_V H(V) = t \log det(tA^{-1}) - tp.$$

(b) Suppose $X_1, X_2, \cdots X_n$ is a random sample from $N_p(\mu, \Sigma)$. Let

$$\begin{aligned} S^2 &= \sum_{i=1}^{n}(X_i - \bar{X})(X_i - \bar{X})' \\ \text{and } S_0^2 &= \sum_{i=1}^{n}(X_i - \mu_0)(X_i - \mu_0)'. \end{aligned}$$

(i) Show that $\hat{\mu} = \bar{X}$ and $\hat{\Sigma} = S^2$ maximise the likelihood function over the whole parametric space.

(ii) Show that $\hat{\Sigma_0} = S_0^2$ maximises the likelihood function over the restricted parametric space in which $\mu = \mu_0$.

(c) Define Hotelling's T^2 and derive its distribution.

(d) Consider the testing of hypothesis problem $H_0 : \mu = \mu_0$ against $H_1 : \mu \neq \mu_0$. Show that the likelihood ratio test statistic is a monotone function of the Hotelling's T^2 statistics. [10 + (6 + 5) + (2 + 5) + 10 = 38]

6. Consider the linear model

$$Y = \mu 1_n + X_T\tau + X_B\beta + \varepsilon,$$

where μ, τ and β are vectors of unknown constants of appropriate order and ε is a random vector with $E(\varepsilon) = 0$ and $Cov(\varepsilon) = \sigma^2 I_n$.

Let $C = (X_B)'(I - P_T)X_B$ and $R = (X_B)'(I - P_T)Y$, where P_T is the projection operator on $\mathcal{C}(X_T)$.

(a) Show that $E(R) = C\beta$ and $Cov(R) = \sigma^2 C$.

(c) Let $SS_B = R'C^- R$. Derive $E[SS_B]$.

(d) Now suppose β is a random vector with $E(\beta) = 0$ and $Cov(\beta) = \sigma_1^2 I_b$.

(i) Find an unbiased estimator of σ^2.

(ii) Show that the expectation of SS_B is of the form $a_1\sigma^2 + a_2\sigma_1^2$, where a_1 and a_2 are functions of X_T and X_B.

[(3 + 4) + 8 + (8 + 10) = 33]